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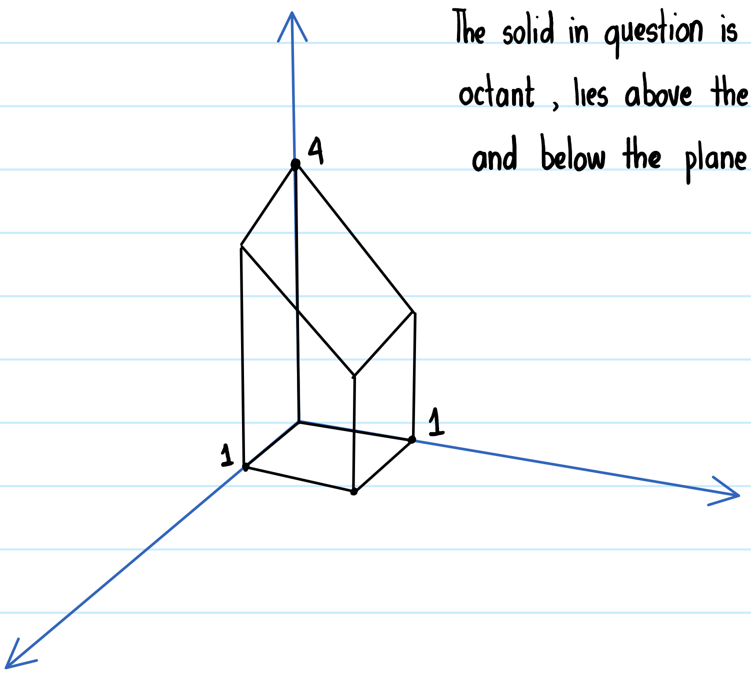
1. $\iint_R \frac{x}{1+xy}\, dA$

$$= \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$$. Let $u = 1+xy$, $du = x\,dy$ [Since we are doing partial integration, at this point, we treat x as a constant]

$$= \int_0^1 \ln|1+xy| \Big]_0^1 dx$$

$$= \int_0^1 \ln|1+x|\, dx = (1+x)\ln|1+x| - (1+x)\Big]_0^1 = 2\ln(2) - 2 - (-1)$$
$$= 2\ln 2 - 1.$$

2) $\int_0^1 \int_0^1 4 - x - 2y\, dy\, dx$



The solid in question is in the first octant, lies above the square $[0,1]\times[0,1]$ and below the plane $z = 4-x-2y$.

3) Recall that area of a domain D is given by $\iint_D 1\, dA$.

In our case, we have a Type 1 domain, so

$$A(D) = \int_1^2 \int_{\ln x}^{x^2} 1\, dy\, dx = \int_1^2 [y]_{\ln x}^{x^2} dx = \int_1^2 x^2 - \ln x\, dx$$

$$= \frac{x^3}{3} - (x\ln x - x)\Big]_1^2 = \frac{2^3}{3} - (2\ln 2 - 2) - \left(\frac{1}{3} - (\ln 1 - 1)\right)$$

$$= \frac{7}{3} - 2\ln 2 + 1 = \frac{10}{3} - 2\ln 2$$

4)

$$f_{ave} = \frac{1}{A(R)} \int_0^{\pi} \int_0^{1} xy\sin(x^2y)\, dx\, dy$$

First compute $\int_0^1 xy\sin(x^2y)\, dx$ (Again, remember y is treated as a constant).

Let $u = x^2y$ when $x = 0,\ u = 0$

$\frac{du}{dx} = 2xy$ $\quad x = 1,\ u = y$

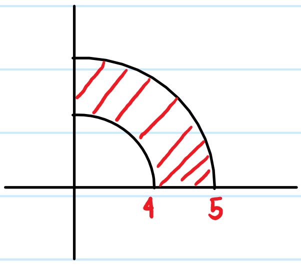
So we get

$$\frac{1}{2}\int_0^y \sin u\, du = \frac{1}{2}\Big[-\cos u\Big]_0^y = \frac{1}{2}\Big[-\cos y + 1\Big]$$

Since R is a rectangle, $A(R) = 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$

So, $f_{ave} = \frac{1}{\pi/2}\int_0^{\pi/2} \frac{1}{2}[1 - \cos y]\, dy = \frac{1}{\pi}\Big[y - \sin y\Big]_0^{\pi/2} = \frac{1}{\pi}\left(\frac{\pi}{2} - 1\right)$

6)



As you can see the domain is a polar rectangle

$R = \{(r,\theta) \mid 4 \le r \le 5,\ 0 \le \theta \le \pi/2\}$

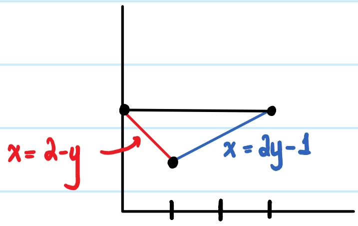
Therefore we can rewrite integral using $x = r\cos\theta$ and $x^2+y^2 = r$ as

$$\int_0^{\pi/2}\int_4^5 (r\cos\theta + r)\, r\, dr\, d\theta = \int_0^{\pi/2}\int_4^5 r^2(1+\cos\theta)\, dr\, d\theta$$

$$= \int_0^{\pi/2}(1+\cos\theta)\, d\theta \int_4^5 r^2\, dr = \Big[\theta + \sin\theta\Big]_0^{\pi/2} \left[\frac{r^3}{3}\right]_4^5$$

$$= \left[\frac{\pi}{2} + 1\right]\left[\frac{125}{3} - \frac{64}{3}\right] = \frac{61}{3}\left(\frac{\pi}{2} + 1\right)$$

6)



Type 2 region

$D = \{(x,y) \mid 1 \le y \le 2,\ 2-y \le x \le 2y-1\}$

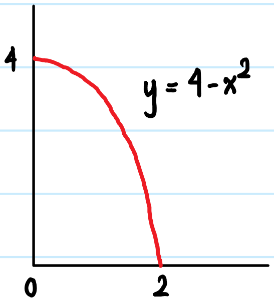
Then, $\iint_D y^3\, dA = \int_1^2 \int_{2-y}^{2y-1} y^3\, dx\, dy = \int_1^2 \Big[xy^3\Big]_{2-y}^{2y-1} dy =$

$$= \int_1^2 y^3\big[2y-1-(2-y)\big]\, dy = \int_1^2 y^3(3y-3)\, dy = \int_1^2 3y^4 - 3y^3\, dy$$

$$= \left[\frac{3y^5}{5} - \frac{3y^4}{4}\right]_1^2 = \left[\frac{96}{5} - \frac{48}{4}\right] - \left[\frac{3}{5} - \frac{3}{4}\right] = \frac{93}{5} - \frac{45}{4}$$

7) $\displaystyle\int_0^2 \int_0^{4-x} \frac{xe^{2y}}{4-y}\, dy\, dx$.

The domain we are integrating over is



Now rewriting the domain as Type II region, we see

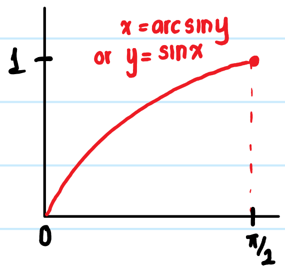
$$D = \{ (x,y) \mid 0 \le y \le 4,\ 0 \le x \le \sqrt{4-y} \}$$

Then,

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy = \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2} \right]_0^{\sqrt{4-y}} dx$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \cdot \frac{4-y}{2}\, dx = \int_0^4 \frac{e^{2y}}{2}\, dy = \frac{1}{2} \frac{e^{2y}}{2} \Big]_0^4 = \frac{e^8 - 1}{4} .$$

8)



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\, dy\, dx$$

$$= \int_0^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, [y]_0^{\sin x}\, dx = \int_0^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, \sin x\, dx$$

Let $u = \cos x$, $du = -\sin x\, dx$, $x = 0,\ u = 1$

$x = \pi/2,\ u = 0$

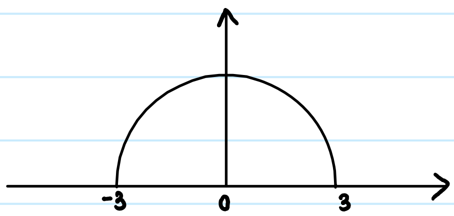
$$= \int_1^0 -u\sqrt{1+u^2}\,du$$

Let $w = 1+u^2$, $dw = 2u\,du$, $u = 1,\ w = 2$; $u = 0,\ w = 1$

$$= \frac{1}{2}\int_2^1 -(w)^{1/2}\,dw = -\frac{1}{3}w^{3/2}\Big]_2^1 = -\frac{1}{3} - \left(-\frac{\sqrt{8}}{3}\right) = \frac{\sqrt{8}-1}{3}$$

10) Example 4, Chapter 15.4.

11) $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$. The domain over which we are integrating is drawn below.



This domain is a polar rectangle

$$R = \{(\theta, \pi) \mid 0 \le \theta \le \pi,\ 0 \le r \le 3\}$$

Therefore,

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^3\int_0^{\pi} \sin(r)\,r\,d\theta\,dr$$

$$= \int_0^3 r\sin r\,[\theta]_0^{\pi}\,dr = \pi\int_0^3 r\sin r\,dr$$

$\longrightarrow$ Integration by parts

$u = r$ $\quad dv = \sin r\,dr$

$du = dr$ $\quad v = -\cos r$

$$= \pi\left[-r\cos r - \int -\cos r\,dr\right]_0^3$$

$$= \pi\left[-r\cos r + \sin r\right]_0^3 = \pi\left[-3\cos 3 + \sin 3\right]$$

13)

D_a is a polar rectangle, $\{(r,\theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq \pi\}$

Then,

$$\iint_{D_a} e^{-(x^2+y^2)}\, dA = \int_0^a \int_0^{2\pi} e^{-r^2} r\, d\theta\, dr = \int_0^a e^{-r^2} r\, [\theta]_0^{2\pi}\, dr$$

$$= 2\pi \int_0^a re^{-r^2} dr = \frac{2\pi}{-2}\left[e^{-r^2}\right]_0^a = -\pi\left(e^{-a^2} - 1\right) = \pi\left(1 - e^{-a^2}\right).$$

Then, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right)$$

$$= \pi$$

b) $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA \Rightarrow$$

$$\Rightarrow \pi = \lim_{a\to\infty}\left[\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\, dy\, dx\right]$$

$$= \lim_{a\to\infty}\left[\int_{-a}^{a}\int_{-a}^{a} e^{-x^2} \cdot e^{-y^2}\, dy\, dx\right]$$

$$\pi = \lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2} dx \cdot \int_{-a}^{a} e^{-y^2} dy\right] = \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2} dx \cdot \lim_{a\to\infty}\int_{-a}^{a} e^{-y^2} dy$$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy$$

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c) $\pi = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$

$\Rightarrow \pi = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-x^2} dx$

$\Rightarrow \pi = \left[\int_{-\infty}^{\infty} e^{-x^2} dx \right]^2$ and since $e^{-x^2} \geq 0$ for all real x, we see that

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

d) $\int_{-\infty}^{\infty} e^{-x^2} dx$. Let $t = \sqrt{2}x$.
$dt = \sqrt{2}\,dx$

Since $\lim_{x \to \infty} t = \lim_{x \to \infty} \sqrt{2}x = \infty$ & $\lim_{t \to -\infty} t = \lim_{t \to -\infty} \sqrt{2}x = -\infty$, we

have

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2} dt \Rightarrow$$

$$\Rightarrow \sqrt{\pi} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2/2} dt \Rightarrow \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$$

$\square$